Acoustic Wave Propagation in Horizontally Variable Media

Cathy Ann Clark, *Member, IEEE*

***Abstract*—An overview of the multipath expansion method of solving the Helmholtz wave equation to describe the underwater sound field for a fixed point source in a plane multilayered medium is presented. The approach is then extended to account for horizontal variations in bottom depth, bottom type, and sound speed in the stationary phase approximation. Comparisons of model results to a limited number of measured data sets and standard propagation codes are presented.**

***Index Terms*—Acoustics, multipath expansion, stationary phase, underwater sound propagation, wave equation.**

I. Introduction

THIS paper presents an overview of the derivation of the propagation loss model of Leibiger [1], [2], and then extends this result to shallow water environments by accounting for changes in bottom depth, bottom type, and sound speed with range. Comparisons between calculated and measured results are presented.

A derivation based in part on the work of Leibiger was published by Weinberg in 1975 [3], [4]. Recent developments by Weinberg extend the use of ray methods through the use of Gaussian ray bundles [5] but do not utilize the integration method discussed in this paper. A comparison of results for a transmission loss (TL) calculation at 3500 Hz using the Integrated Mode method described herein and Weinberg's Navy standard Gaussian RAy Bundling (GRAB) model is also presented.

The calculation of sound propagation in the low frequency portion of the sonic band (20 to 200 Hz) has been addressed using ray theory, parabolic equation (PE) methods, and normal mode codes, with which this approach agrees. However, the calculation method described herein is also viable at higher frequencies, from 200 Hz to above 50 kHz. The need for calculations which are continuous across the operational band of Navy platforms and systems in shallow water provides motivation for the current work.

An overview of the Leibiger derivation is presented in Section III, followed by an extension to range-varying media in Section IV. Section V provides a comparison of results to a limited number of measured data sets and other propagation codes.

Manuscript received April 15, 2003; revised June 15, 2004; accepted July 15, 2004. Associate Editor: W. Carey.

The author is with the Naval Undersea Warfare Center, Newport, RI 02841 USA.

Digital Object Identifier 10.1109/JOE.2004.834167

II. Preliminaries

We define sound speed c to take values within the vertical plane of the ocean defined by the positions of the source and receiver of the acoustic energy. We then define the total wavenumber κ to be equal to ω/c, where ω is the radian frequency of the source and denote the horizontal and vertical vector components of the wavenumber κ by k_r and k_z, respectively, where r denotes range and z denotes depth. We define θ to be the angle of the total wavenumber with the horizontal, and denote the vertical phase function ξ as

$$\xi = \int_z k_z dz. \tag{1}$$

Assuming a region which is homogeneous in range, we evaluate ξ and its derivatives by partitioning the local sound speed profile (SSP) into linear segments in the usual way, thus defining a horizontally stratified ocean. In horizontally variable media, we likewise partition the range of interest into segments for which 1) sound speed variation with depth; 2) bottom slope; and 3) bottom type are considered invariant. Thus, calculations are always performed within a rectangle in which the sound speed variation is the same linear function of depth throughout the region.

The horizontal range traversed in propagating through a layer from z_1 to z_2 with constant sound speed gradient g is given by

$$\Delta r = -\frac{c_v}{g}\Delta \sin\theta \tag{2}$$

where c_v denotes the vertex sound speed at which all propagation is momentarily horizontal $(\theta(z) = 0)$.

Substituting $k_z = \sqrt{(\omega/c)^2 - k_r^2}$ into (1), using the fact that $k_r = (\omega/c_v)$, performing a change of variable $dz = (1/g)dc$, and integrating yields

$$\begin{aligned}\xi = \int k_z dz &= \int \sqrt{\left(\frac{\omega}{c}\right)^2 - \left(\frac{\omega}{c_v}\right)^2} dz \\ &= \frac{\omega}{c_v g}\int \frac{\sqrt{c_v^2 - c^2}}{c} dc \\ &= -\frac{1}{g}\left(ck_z + \omega \ln \frac{k_r}{\omega/c + k_z}\right).\end{aligned} \tag{3}$$

Thus the change in vertical phase in traversing a horizontal layer from z_1 to z_2 is computed as

$$\Delta\xi = \xi(c(z_2) - \xi(c(z_1)). \tag{4}$$

Report Documentation Page

Form Approved
OMB No. 0704-0188

1. REPORT DATE **2005**	2. REPORT TYPE	3. DATES COVERED **00-00-2005 to 00-00-2005**
4. TITLE AND SUBTITLE **Acoustic Wave Propagation in Horizontally Variable Media**		5a. CONTRACT NUMBER 5b. GRANT NUMBER 5c. PROGRAM ELEMENT NUMBER
6. AUTHOR(S)		5d. PROJECT NUMBER 5e. TASK NUMBER 5f. WORK UNIT NUMBER
7. PERFORMING ORGANIZATION NAME(S) AND ADDRESS(ES) **Naval Undersea Warfare Center,Newport Division,Newport,RI,02841**		8. PERFORMING ORGANIZATION REPORT NUMBER
9. SPONSORING/MONITORING AGENCY NAME(S) AND ADDRESS(ES)		10. SPONSOR/MONITOR'S ACRONYM(S) 11. SPONSOR/MONITOR'S REPORT NUMBER(S)

12. DISTRIBUTION/AVAILABILITY STATEMENT

Approved for public release; distribution unlimited

13. SUPPLEMENTARY NOTES

14. ABSTRACT

An overview of the multipath expansion method of solving the Helmholtz wave equation to describe the underwater sound field for a fixed point source in a plane multilayered medium is presented. The approach is then extended to account for horizontal variations in bottom depth, bottom type, and sound speed in the stationary phase approximation. Comparisons of model results to a limited number of measured data sets and standard propagation codes are presented.

15. SUBJECT TERMS

16. SECURITY CLASSIFICATION OF:			17. LIMITATION OF ABSTRACT	18. NUMBER OF PAGES	19a. NAME OF RESPONSIBLE PERSON
a. REPORT **unclassified**	b. ABSTRACT **unclassified**	c. THIS PAGE **unclassified**	**Same as Report (SAR)**	**10**	

Standard Form 298 (Rev. 8-98)
Prescribed by ANSI Std Z39-18

Note that

$$\frac{d}{dk_r}k_z(z,k_r) = \frac{d}{dk_r}\sqrt{\left(\frac{\omega}{c}\right)^2 - k_r^2} = -k_r/k_z \quad \text{and so}$$

differentiating (3) with respect to k_r yields

$$\frac{d\xi}{dk_r} = -\frac{1}{g}\left(\left(c - \omega\frac{1}{\omega/c + k_z}\right)\frac{dk_z}{dk_r} + \frac{\omega}{k_r}\right) = -\frac{1}{g}\frac{ck_z}{k_r}.$$

Multiplying and dividing by c_v and utilizing Snell's Law yields

$$\frac{d\xi}{dk_r} = -\frac{c_v}{g}\frac{c}{c_v}\frac{k_z}{k_r} = -\frac{c_v}{g}\sin\theta.$$

For use in the sequel, we note that

$$\frac{d}{dk_r}\int_{z_1}^{z_2} k_z dz = \int_{z_1}^{z_2}\frac{dk_z}{dk_r}dz = -\int_{z_1}^{z_2}\frac{k_r}{k_z}dz. \tag{5}$$

III. Overview of the Leibiger Derivation

We now give an overview of the Leibiger derivation in preparation for extending to horizontally variable media; a complete discussion of the original work may be found in [1]. Derivations of the acoustic wave equation and further discussion of various solutions can also be found in multiple references, see, for example [6]–[9].

Assuming sound pressure φ has no azimuthal dependence, the wave equation can be expressed in cylindrical coordinates as

$$\frac{\partial^2\varphi}{\partial r^2} + \frac{1}{r}\frac{\partial\varphi}{\partial r} + \frac{\partial^2\varphi}{\partial z^2} = \frac{1}{c^2}\frac{\partial^2\varphi}{\partial t^2} \tag{6}$$

where r represents horizontal range, and depth z is increasing down from the ocean surface. After a standard separation of variables, the depth dependent function $Z(z)$ is given by

$$Z''(z) + k_z^2(z)Z(z) = \delta(z - z_0) \tag{7}$$

where $k_z(z)$ is the vertical component of the local wavenumber, and we assume a harmonic point source of unit intensity at depth z_0. We construct solutions of the depth dependent wave equation from the Green's function for the associated boundary value problem (see, for example, [11], [12]). At this point, we drop the subscript r from the horizontal wavenumber to simplify the notation, i.e., in the sequel, the unsubscripted variable k implies k_r. Let $z = \alpha$ be a turning point for the wavenumber k, thus $k_z^2(\alpha, k) = 0$. We assume that k_z^2 can be expressed as a linear function of z in a neighborhood of α, and let

$$k_z^2(z,k) = M^2(z-\alpha) + o(z-\alpha)$$

where M is a positive constant and both k and α are assumed real. Integrating from α to an arbitrary depth $z > \alpha$ to obtain the elapsed vertical phase $\xi(z,k)$ yields

$$\xi(z,k) = \int_\alpha^z k_z(z,k)dz = \frac{2}{3}M(z-\alpha)^{3/2} + \dots. \tag{8}$$

For $k_z^2 < 0$ we maintain positive values for phase by adopting the convention $k_z = -i|k_z|, \xi = i|\xi|$. Substitution of the leading term of ξ from (8) into the homogeneous form of the z-dependent wave (7), letting $Z = (z-\alpha)^{1/2}y$, and changing the independent variable to ξ yields Bessel's equation of order 1/3 [10]

$$\frac{d^2y}{d\xi^2} + \frac{1}{\xi}\frac{dy}{d\xi} + \left(1 - \frac{1}{9\xi^2}\right) = 0. \tag{9}$$

Thus cylinder functions of order $1/3$ provide solutions for Z, for example,

$$Z = (z-\alpha)^{1/2}J_{1/3}(\xi). \tag{10}$$

Noting that $z - \alpha$ is proportional to ξk_z^{-1}, a set of allowable solutions for propagation toward and away from α when compared with time dependence $e^{i\omega t}$ are given by

$$\begin{aligned} Z_1 &= K_1\sqrt{\frac{\xi}{k_z}}H_{1/3}^{(1)}(\xi) \quad \text{and} \\ Z_2 &= K_2\sqrt{\frac{\xi}{k_z}}H_{1/3}^{(2)}(\xi) \end{aligned} \tag{11}$$

where $k_z^2 > 0$ is assumed and K_1 and K_2 are suitable constants.

The leading terms of the asymptotic expansions of $H_{1/3}(\xi)$ and $H_{1/3}^{(2)}(\xi)$ when $\arg(\xi) > -\pi$ are given by [10]

$$\begin{aligned} H_{1/3}^{(1)}(\xi) &\sim \sqrt{\frac{2}{\pi\xi}}e^{i(\xi - 5\pi/12)} \\ H_{1/3}^{(2)}(\xi) &\sim \sqrt{\frac{2}{\pi\xi}}e^{-i(\xi - 5\pi/12)}. \end{aligned} \tag{12}$$

Substitution of (12) into (11) yields, with the appropriate choice of constants

$$\begin{aligned} Z_1 &\sim \sqrt{\frac{2}{\pi}}e^{-i5\pi/12}\frac{1}{\sqrt{k_z}}e^{i\xi} \quad \text{and} \\ Z_2 &\sim \sqrt{\frac{2}{\pi}}e^{-i\pi/12}\frac{1}{\sqrt{k_z}}e^{-i\xi} \end{aligned} \tag{13}$$

which are valid away from the turning point $z = \alpha$ on the side where $|\cos\theta| < 1$ (they become infinite when $k_z^2 = 0$). The Bessel function expressions (11) allow the solution of the differential equation to be approximated over a specific z-interval, both near and away from α, and they agree with the exponential solutions (12) in regions where they are oscillatory.

To simplify the notation, we choose the constants K_1 and K_2 so that the depth dependent function is simply

$$\begin{aligned} Z_1 &= ve^{i\xi} \\ Z_2 &= ve^{-i\xi} \end{aligned} \tag{14}$$

where $v(z,k) \sim (1/\sqrt{k_z})$, and determine the linear combination of solutions Z_1 and Z_2 which solves the boundary value problem in the z-domain.

For the purpose of this paper we defer the discussion of rigorous formulations for the boundary conditions and assume that the reflection loss and phase change at the upper (z_1) and lower

(z_2) boundaries can be characterized as a function of grazing angle θ by complex reflection coefficients

$$R_1 = |R_1|e^{i\Phi_1} \quad \text{and}$$
$$R_2 = |R_2|e^{i\Phi_2} \tag{15}$$

respectively, and thus that the fundamental wave functions u_1 and u_2 satisfy the upper and lower boundary conditions, respectively, i.e.

$$u_1 = Z_1 + R_1 Z_2$$
$$u_2 = Z_2 + R_2 Z_1.$$

Substituting the fundamental solutions (14), then, we have

$$u_1 = v(z,k)\left(e^{-i\int_z^{z_1} k_z dz} + R_1 e^{+i\int_z^{z_1} k_z dz}\right) \quad \text{and}$$
$$u_2 = v(z,k)\left(e^{+i\int_z^{z_2} k_z dz} + R_2 e^{-i\int_z^{z_2} k_z dz}\right). \tag{16}$$

We form the Green's function solution from the upper and lower fundamental solutions

$$Z(z,z_0,k) = \begin{cases} \frac{u_1(z)u_2(z_0)}{W(k)}, & z < z_0 \\ \frac{u_2(z)u_1(z_0)}{W(k)}, & z > z_0 \end{cases} \tag{17}$$

where z_0 is the depth of the source and $W(k)$ is the Wronskian

$$W(k) = u_1(z)u_2'(z) - u_2(z)u_1'(z). \tag{18}$$

The Green's function construction extended to the range domain yields the pressure φ as a contour integral in the complex k-plane

$$\varphi(r,z) = 2\oint H_0^{(2)}(kr)Z(z,z_0,k)kdk \tag{19}$$

where the integral encompasses all of the zeros of the Wronskian.

We develop a practical expression for $\varphi(r,z)$ by formulating the Wronskian $W(k)$ from the basic asymptotic forms of $u_1(z,k)$ and $u_2(z,k)$ given in (16), which, after substituting into (18), and performing some algebra, yields

$$W(k) = -2ie^{\int_{z_1}^{z_2} k_z dz} W_0(k) \tag{20}$$

where

$$W_0(k) = 1 - |R_1||R_2|e^{i(\Phi_1+\Phi_2-2\int_{z_1}^{z_2} k_z dz)}. \tag{21}$$

The normal modes, which correspond to the zeros of $W_0(k)$, take sequential values $k = k_n, n = 1, 2, \ldots$ where n is the mode number. In general, the k_n are complex with small imaginary part

$$k_n = \mathrm{Re}(k_n) + i\varepsilon_n, \quad \varepsilon_n < 0, \quad |\varepsilon_n| \ll |\mathrm{Re}(k_n)|.$$

To extract the roots of $W_0(k)$, consider first a propagation medium in which boundary reflections are without loss, i.e., let $|R_1| = |R_2| = 1$. In this special case, the zeros of $W_0(k)$ are given by real values of k for which the total phase change in traversing the medium in depth from upper turning point to upper turning point is an integer multiple of 2π, i.e.,

$$\int_{z_1}^{z_2} k_z dz - \Phi_1 - \Phi_2 = 2n\pi, \quad n = 1, 2, \ldots. \tag{22}$$

The nonidealized ocean propagation medium is lossy, requiring $|R_1 R_2| < 1$. This leads to nonzero imaginary parts ε_n of the eigenvalues k_n, which are found by expansion of (22) about the real part of k. To expand $k_z(z,k^2)$ around $\mathrm{Re}(k^2)$ to the first order

$$\frac{d}{dk}k_z(z,k^2) = \frac{d}{dk}\sqrt{(\omega/c)^2 - k^2}$$
$$= \frac{-2k}{2\sqrt{(\omega/c)^2 - k^2}} = \frac{-k}{k_z(z,k^2)}$$

so

$$k_z(z,k^2) = k_z(z,\mathrm{Re}(k^2)) - \frac{\mathrm{Re}(k)}{k_z(z,\mathrm{Re}(k^2))}(k - \mathrm{Re}(k)).$$

But $k - \mathrm{Re}(k) = i\varepsilon$, so we have

$$1 - R_1 R_2 e^{-2i\int_{z_1}^{z_2}\left(k_z(z,\mathrm{Re}(k^2)) - \frac{\mathrm{Re}(k)i\varepsilon}{k_z(z,\mathrm{Re}(k^2))}\right)dz} = 0, \quad \text{or}$$
$$1 - R_1 R_2 e^{-2i\int_{z_1}^{z_2} k_z(z,\mathrm{Re}(k^2))dz} e^{-2\varepsilon\int_{z_1}^{z_2}\frac{\mathrm{Re}(k)}{k_z(z,\mathrm{Re}(k^2))}dz} = 0.$$

Combining this result with (5), the imaginary part of the eigenvalue for mode n is then given by

$$\varepsilon_n = \frac{\ln(|R_1 R_2|)}{R_c(\mathrm{Re}(k_n))} \tag{23}$$

as can be verified by substitution of the above into the expression for $W_0(k)$ (21).

Although some variations of the boundary value problem may include propagation of an infinite number of normal modes in the ocean medium, in practical applications a possibly large but finite number of modes suffices to produce $\varphi(r,z)$. Evalulating by the calculus of residues yields

$$\varphi(r,z) = -i\pi\sum_{n=1}^{N} H_0^{(2)}(kr)G(z,k)k\, e^{-i\int_{z_1}^{z_2} k_z dz}\Big|_{k=\mathrm{Re}(k_n)} \tag{24}$$

where $G(z,k) = (u_1(z,k)u_2(z,k)/R_c(k))$. Note that all expressions in the above sum may be evaluated using the real part of the eigenvalue with the exception of the range dependent Hankel function in which the imaginary ε_n provides an exponential attenuation factor $e^{\varepsilon_n r}$ which accounts for the boundary losses $|R_1|$ and $|R_2|$.

An equivalent expansion for $\varphi(r,z)$ is obtained by noting that

$$\frac{1}{W_0(k)} = 1 + \sum_{q=1}^{\infty} |R_1R_2|^q e^{iq(\Phi_1+\Phi_2-2\int_{z_1}^{z_2} k_z dz)}. \qquad (25)$$

After substitution of (25) into (19), we note that the major contribution to the contour integral occurs within a specific interval (k_1, k_2) on the real axis (see, for example, [13, ch. 4]). After some algebraic manipulation, the qth term of the resulting series expansion for $\varphi(r,z)$ is

$$i\int_{k_1}^{k_2} H_0^{(2)}(kr)u_1(z)u_2(z)|R_1R_2|^q e^{iq(\Phi_1+\Phi_2-2\int_{z_1}^{z_2} k_z dz)} k\, dk. \qquad (26)$$

Note also that the finite integration interval corresponds to the use of only N modes of a possibly infinite set in the normal mode sum and each term corresponds to a specific propagation path in the ocean.

Separation of the qth integral into four path types is accomplished by substituting the fundamental wave solutions u_1 and u_2 from (16) into (26) and noting that they are linearly dependent at an eigenvalue. To simplify the notation, denote the phase exponents as

$$\xi_c = \int_{z_1}^{z_2} k_z dz,$$
$$\xi_{r1} = \int_{z_1}^{z} k_z dz,$$
$$\xi_{r2} = \int_{z_2}^{z} k_z dz$$
$$\xi_s = \int_{z_1}^{z_0} k_z dz, \quad \text{and}$$
$$\xi_{sr} = \int_{z}^{z_0} k_z dz.$$

Neglecting amplitude terms, then

$$\begin{aligned} &u_1(z)u_2(z_0)e^{-i\xi_c} \\ &\quad = (e^{-i\xi_r} + R_2e^{i\xi_r})(e^{-\xi_s} + R_1e^{-i\xi_s})e^{-i\xi_c} \\ &\quad = e^{-i\xi_c}[e^{-\xi_c}e^{-i\xi_{r1}} + R_2e^{-i\xi_c}e^{i\xi_{r1}}](e^{i\xi_s} + R_1e^{-i\xi_s}) \\ &\quad = [e^{-i\xi_{r1}} + R_2e^{-2i\xi_c}e^{i\xi_{r1}}](e^{i\xi_s} + R_1e^{-i\xi_s}). \end{aligned}$$

But $R_2e^{-2i\xi_c} = 1/R_1$ so we have

$$u_1(z)u_2(z_0)e^{-i\xi_c} = (e^{-i\xi_{r1}} + 1/R_1e^{i\xi_{r1}})(e^{i\xi_s} + R_1e^{-i\xi_s}).$$

The result of multiplying the two factors yields the four principal terms

$$\underset{(1)}{e^{-i\xi_{sr}}} + \underset{(2)}{R_1e^{-i\xi_{r1}}e^{-i\xi_s}} + \underset{(3)}{\frac{1}{R_1}e^{i\xi_{r1}}e^{i\xi_s}} + \underset{(4)}{e^{i\xi_{sr}}}. \qquad (27)$$

These four terms correspond to the four possible combinations of vertical path direction at the source and receiver, i.e., path (1) corresponds to a downward source angle and upward receiver angle, etc. The path correspondence is summarized in Table I.

TABLE I
MULTIPATH/ANGLE CORRESPONDENCE

Path	Source Angle	Receive Angle
(1)	down	up
(2)	up	up
(3)	down	down
(4)	up	down

Evaluating (26) by the method of stationary phase [13], we note that for fixed receiver depth z_r, each integral evaluated at range r is of the form

$$I(r) = \int_{k_1}^{k_2} g(k)e^{-if(k)}dk \qquad (28)$$

where $g(k)$ is slowly varying in comparison to $e^{-if(k)}$, and $f(k)$ has continuous derivatives to the second order. Thus $f(k)$ can be expanded about an arbitrary point k^* within the integration interval

$$f(k) = f(k^*) + f'(k^*)(k-k^*) + 1/2f''(k^*)(k-k^*)^2 + \ldots$$

If a stationary phase point k^* with $f'(k^*) = 0$ can be found within the interval, (28) simplifies to

$$I(r) = g(k^*)e^{-if(k^*)}\int_{k_1}^{k_2} e^{-i/2f''(k^*)(k-k^*)^2}dk. \qquad (29)$$

Assuming $f''(k^*) > 0$, the change of variable $x = \sqrt{1/2f''(k^*)}(k-k^*)$ yields

$$I(r) = \frac{g(k^*)e^{-if(k^*)}}{\sqrt{\frac{f''(k^*)}{2}}}\int_{x_1}^{x_2} e^{-ix^2}dx \qquad (30)$$

which can be integrated numerically. In cases where the integration limits can be considered infinite (i.e., in intervals containing a stationary phase point k^* and integration limits sufficiently far from k^*), the integration can be eliminated by noting that

$$\int_{-\infty}^{+\infty} e^{\pm ix^2}dx = e^{\pm i\pi/4}\sqrt{\pi}. \qquad (31)$$

If $f''(k^*) < 0$, the solution is the complex conjugate of that for $f''(k^*) > 0$. In some cases the solution is extrapolated into diffraction regions, and when there is no stationary phase point within the interval, integration by parts is used to approximate the basic integral.

IV. EXTENSION TO HORIZONTALLY VARIABLE MEDIA

The extension of the results of Section III to media in which environmental properties may vary with range is accomplished in two stages. The first generalizes the calculation of the vertical phase function (1) and its derivatives to accommodate horizontal changes in bottom slope, sound speed, and bottom type and the second accounts for the change in horizontal wavenumber with range.

A. Generalization of Vertical Phase Calculations

Following the notation of Section III (28), we generalize the phase calculations to account for horizontal variations in bottom depth, bottom type, and sound speed. As before, $f'(k)$ corresponds to range, and $f''(k)$ to the derivative of range with respect to horizontal wavenumber k, which can be approximated locally by $(\Delta r/\Delta k)$.

In Section III [e.g., (25), (26)], boundary losses were raised to a power q which represents cycle number. This assumes the loss on each boundary interaction is the same, and the cycles of each path are homogeneous. When bottom and sound speed parameters vary with range, this may not be the case. The notion of cycle carries over as traversal of a wave from one upper turning point or reflection to the next, but the paths traversed within these cycles may vary.

To account for horizontal variations in sound speed, phase velocities are corrected according to Snell's Law. To account for variable bathymetry, it suffices to detect when the bottom is encountered within a given rectangle, and adjust the ray calculations to account for the local bottom slope. On passing a range at which the bottom type changes, bottom loss calculations utilize the properties of the new bottom type. As rays are traced through the medium, the values of the phase derivatives (4), (2), and $f'' \approx \Delta r/\Delta k$ are accumulated as a function of range and used in the minimum phase expansion of the mode integrals (30).

B. Adjustment for Horizontal Wavenumber Differential

One final adjustment is required to extend the mode integrals (30) into horizontally variable media. Note that when the ray calculations are adjusted for SSP and/or bottom depth changes, a given phase velocity, c_v, or equivalently, horizontal grazing angle, θ, is adjusted in accordance with either Snell's Law or specular reflection, respectively. This adjustment results in association of the given ray path with a different horizontal wavenumber, k.

In terms of the integral form (26), this adjustment necessitates a correction to the integrand to account for the change of variable from the original wavenumber k_0 to the local wavenumber $k(k_0)$. Considering the mode summation (24), which the integral approximates, the change in wavenumber corresponds to a coupling of the normal modes between the two adjacent vertical regions. A discussion to demonstrate the change of variable which accounts for this coupling is as follows.

Consider the integral form (26) which corresponds to the qth term in the series expansion for $\varphi(r,z)$, and assume the asymptotic form of the Hankel function, and exponentials as fundamental solutions to the Green's function (16). Assuming boundary losses (15) are included in the amplitude functions denoted by $v(z,k)$, and denoting the elapsed vertical phase along the ray path Γ including reflection effects by $\xi_\Gamma(k)$, we formulate an individual mode integral for a receiver at range r and depth z_r as

$$I(r) = \int v(z_0,k_0)v(z_r,k_0)e^{-i(\xi_\Gamma(k_0)+k_0 r)}\sqrt{k_0}dk_0 \qquad (32)$$

where the subscript Γ denotes accumulation along the path, and k_0 denotes the horizontal component of wavenumber at the source position.

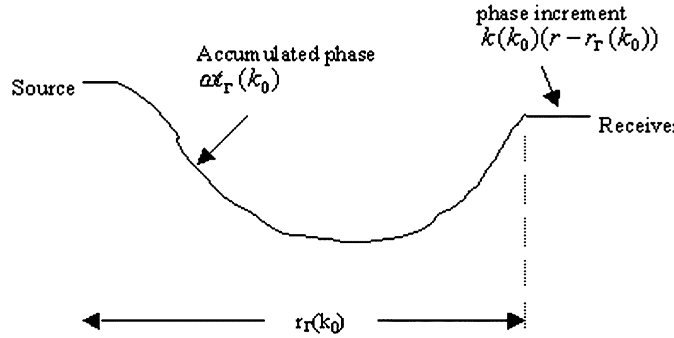

Fig. 1. Sample path for phase integration.

The final factor of $\sqrt{k_0}$ in the integrand is a combination of k_0 and $1/\sqrt{k_0}$ which two factors appear at two different points in the development. The first is from the factor $2k_0dk_0$ which resulted from a change of variable from k_0^2 to k_0 in considering the k-plane equivalent to the contour integral in the k^2 plane (see [1, Section 2.1]), i.e., $d(k_0^2) = 2k_0dk_0$. The second $1/\sqrt{k_0}$ resulted from use of the asymptotic form of the Hankel function. In passing to the range dependent formulation, the first of these factors remains the same, but the second must account for variation in the local wavenumber.

We reformulate the phase integral (3) by equating spatial and temporal phase

$$\omega t = \int k_z dz + kr = \xi(k) + kr \qquad (33)$$

and substitute $\xi_\Gamma(k_0) = \omega t_\Gamma(k_0) - k_0 r_\Gamma(k_0)$ so that (28) becomes

$$I(r) = \int v(z_0,k_0)v(z_r,k_0)\frac{e^{-i\omega t_\Gamma(k_0)}}{\sqrt{k_0}}e^{-i(k_0(r-r_\Gamma(k_0)))}k_0dk_0. \qquad (34)$$

To account for the change in horizontal wavenumber from source to receiver, some of the k_0s must be replaced by $k(k_0)$, the local wavenumber at field point (r,z_r) for the ray with wavenumber k_0 at the source

$$I(r) = \int A(z_0,z_r,k_0)\frac{e^{-i\omega t_\Gamma(k_0)}}{\sqrt{k(k_0)}}e^{-ik(k_0)(r-r_\Gamma(k_0))}k_0dk_0 \qquad (35)$$

where $A(z_0,z_r,k_0) = v(z_0,k_0)v(z_r,k(k_0))$.

The sketch of a sample path in Fig. 1 indicates the components of phase for a receiver at range r. Clearly, the amplitude factors associated with source and receiver are to be evaluated at k_0 and $k(k_0)$, respectively, and the sketch also indicates the use of k_0 and $k(k_0)$ in the phase exponents.

To justify the value $\sqrt{k(k_0)}$ in the denominator, consider the accumulated transmission coefficients across the vertical regions from source to receiver at range r. Consider, for example, a transition at range r_0 from region i to $i+1$ with horizontal wavenumbers k_i and k_{i+1}, respectively, as depicted in Fig. 2. Assuming an incident plane wave of unit amplitude, continuity of the wave function and first derivative at r_0 yields

$$e^{ik_i(r-r_0)} + R_i e^{-ik_i(r-r_0)} = T_i\, e^{ik_{i+1}(r-r_0)}\Big|_{r=r_0} \quad \text{and}$$

$$ik_i e^{ik_i(r-r_0)} - ik_i R_i e^{-ik_i(r-r_0)} = ik_{i+1}T_i\, e^{ik_{i+1}(r-r_0)}\Big|_{r=r_0}$$

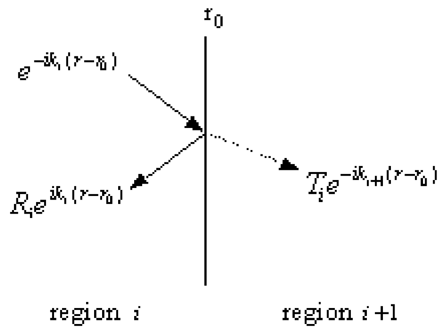

Fig. 2. Horizontal transmission coefficient.

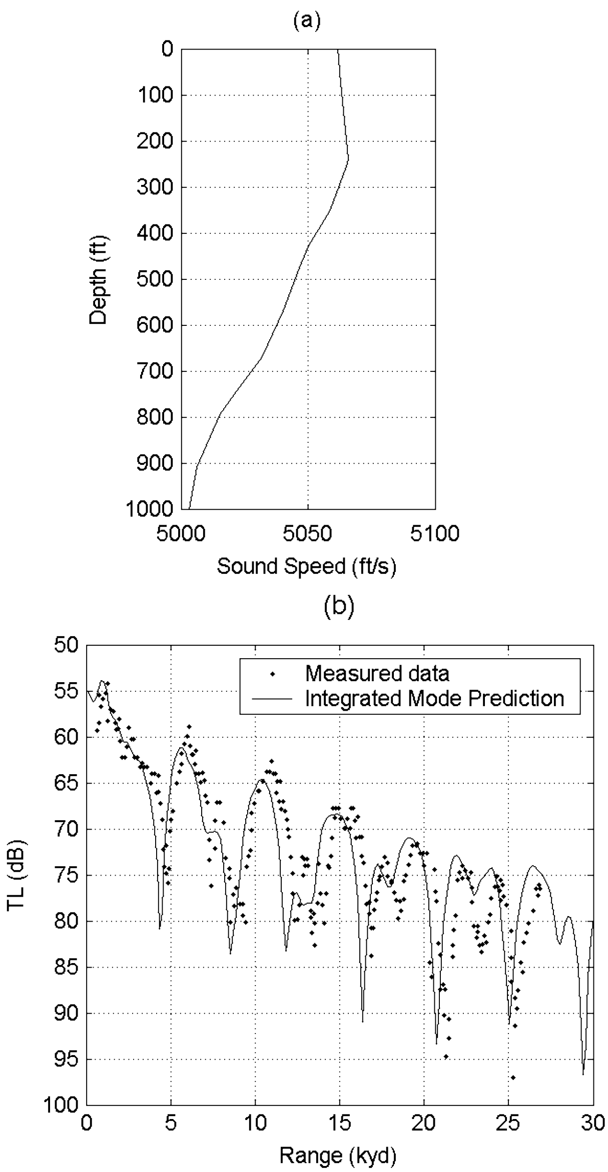

Fig. 3. Example Case A: TL versus range for a 3520-Hz source in a surface duct environment.

which implies

$$1 + R_i = T_i \quad \text{and}$$
$$k_i(1 - R_i) = k_{i+1}T_i.$$

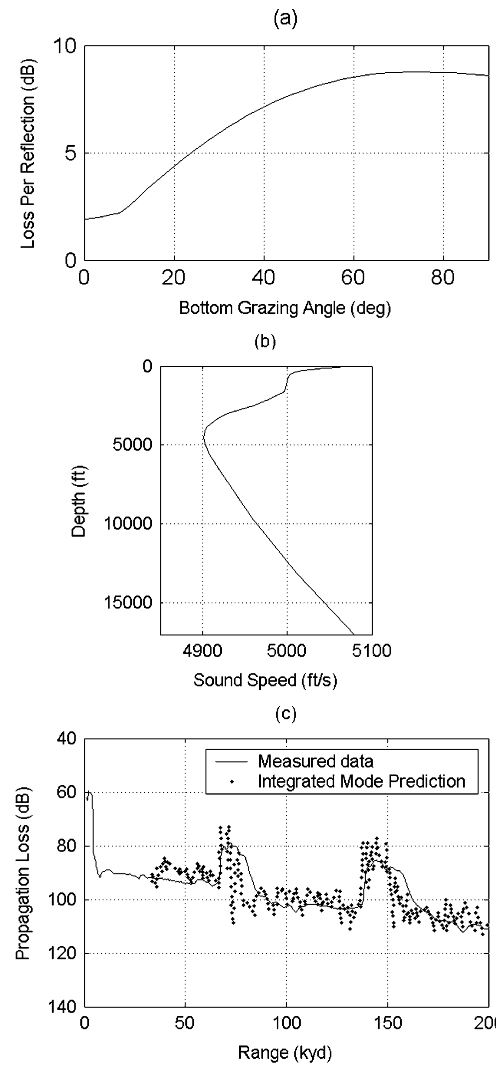

Fig. 4. Example Case B: TL versus range for a 350-Hz source in a convergence zone environment.

Solving for T_i, we obtain the magnitude of the transmission coefficent

$$T_i = \frac{2k_i}{k_i + k_{i+1}}.$$

We perform some algebraic manipulation to approximate T_i in a more convenient form

$$T_i = \frac{2k_i}{k_{i+1} + k_i} = \frac{2}{1 + \frac{k_{i+1}}{k_i}} = \frac{2}{2 + \frac{k_{i+1} - k_i}{k_i}} = \frac{1}{1 + \frac{k_{i+1} - k_i}{2k_i}}.$$

Thus, to first order, we can approximate

$$T_i \approx \frac{1}{\sqrt{1 + \frac{k_{i+1} - k_i}{k_i}}} = \sqrt{\frac{k_i}{k_{i+1}}}.$$

Over N vertical regions, the overall transmission coefficient T is approximated by forming the product

$$T = \prod_{i=0}^{N} T_i \approx \sqrt{\left(\frac{k_0}{k_1}\right)\left(\frac{k_1}{k_2}\right)\cdots\left(\frac{k_{N-1}}{k_N}\right)} = \sqrt{\frac{k_0}{k_N}}.$$

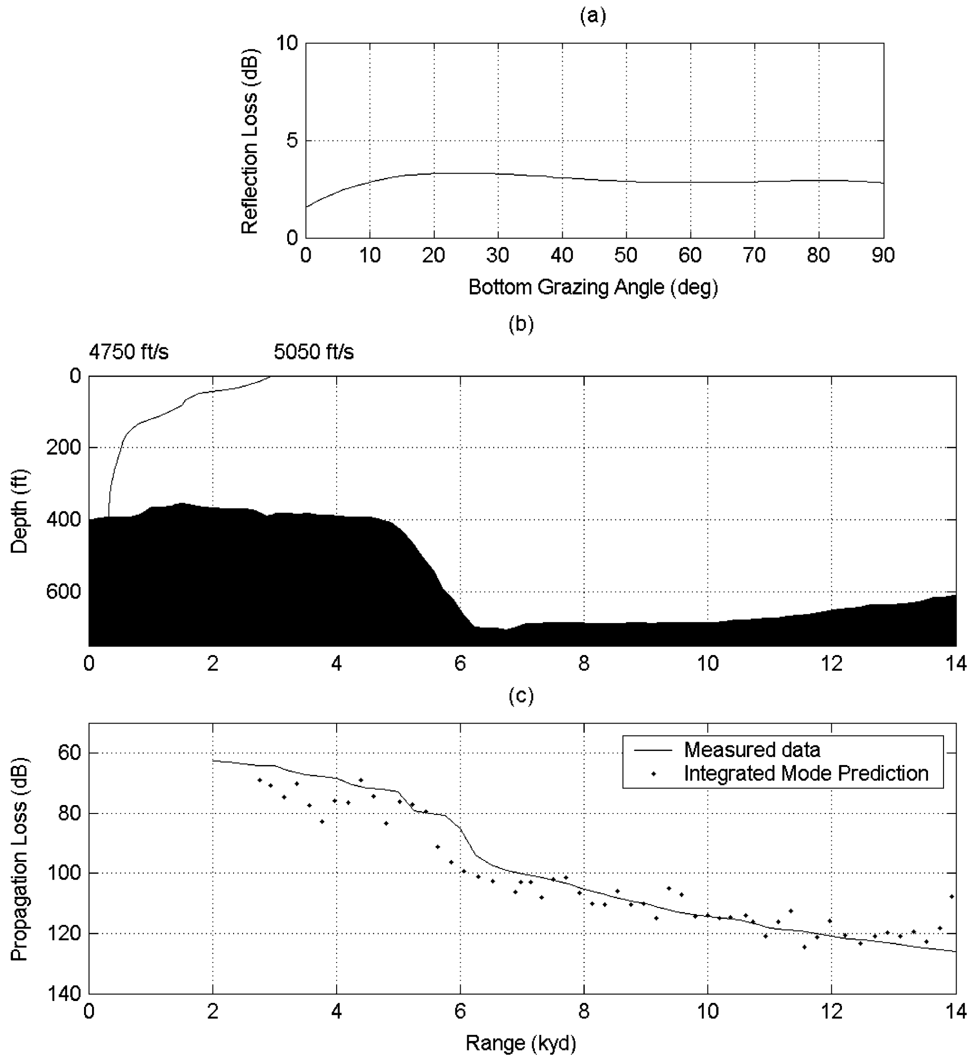

Fig. 5. Example Case C: TL versus range for a 3860-Hz source in shallow water.

Noting that k_N in this case is $k(k_0)$, then, we see that the amplitude of the radial propagation is altered in such a way that the original $\sqrt{k_0}$ in the denominator is replaced by $\sqrt{k(k_0)}$.

To apply the method of stationary phase to the range dependent form of the mode integral (26), following the use of the Fresnel integral form (28), we have

$$I(r) = A(z_0, z_r, k_0^*) \frac{k_0^*}{\sqrt{k(k_0^*)}} \frac{\sqrt{2}}{\sqrt{\frac{dr}{dk_0}}\sqrt{\frac{dk}{dk_0}}} \times e^{-i\Phi(k_0^*, r)} \int e^{-ix^2} dx$$

where $A(z_0, z_r, k_0^*) = v(z_0, k_0^*)v(z_r, k_0^*)$, and $\Phi(k_0^*, r) = (\xi_\Gamma(k_0^*) + k(k_0^*)r$. Noting that $(dk/dk_0) = (\tan\theta_r / \tan\theta_{0r})$, where θ_{0r} and θ_r are the ray angles at the source and receiver, and using the asymptotic form for the amplitude functions

$$v(z_r, k_0^*) = \frac{1}{\sqrt{k_z(z_r, k_0^*)}} = \frac{1}{\sqrt{k_0^* \tan\theta_{0r}}}$$

yields

$$I(r) = v(z_0, k_0^*) \frac{\sqrt{2k_0^*}}{\sqrt{\tan\theta_r k(k_0^*) \frac{dr}{dk_0}}} e^{-i\xi(k_0^*, r)} \int e^{-ix^2} dx.$$

TABLE II
BOTTOM SEDIMENT PARAMETERS FOR EXAMPLE CASE D

Range	0	3	6	7.5	12.09
Sthick	1.091	1.091	1.091	1.091	1.091
Xratio	1.009	1.0146	1.0112	1.0094	1.0015
Xgrad	.634	.634	.634	.634	.634
Beta	25	25	25	25	25
Satten	.06477	.06477	.06477	.06477	.06477
Sagrad	0	0	0	0	0
Sdense	1.8	1.8	.8	1.8	1.8
Xdense	1.8	1.8	1.8	1.8	1.8
Xthick	0	0	0	0	0
Baslos	.069	.069	.069	.069	.069

Finally, recognizing that $(1/\sqrt{k(k_0^*)\tan\theta_r})$ is asymptotically $v(z_r, k)$, we have the required result

$$I(r) = A(z_0, z_r, k(k_0))\sqrt{k_0^*} \frac{\sqrt{2}}{\sqrt{\frac{dr}{dk_0}}} e^{-i\xi(k_0^*, r)} \int e^{-ix^2} dx. \tag{36}$$

This result is the range dependent analog of (30). The amplitude factors v are computed as before but as a function of the

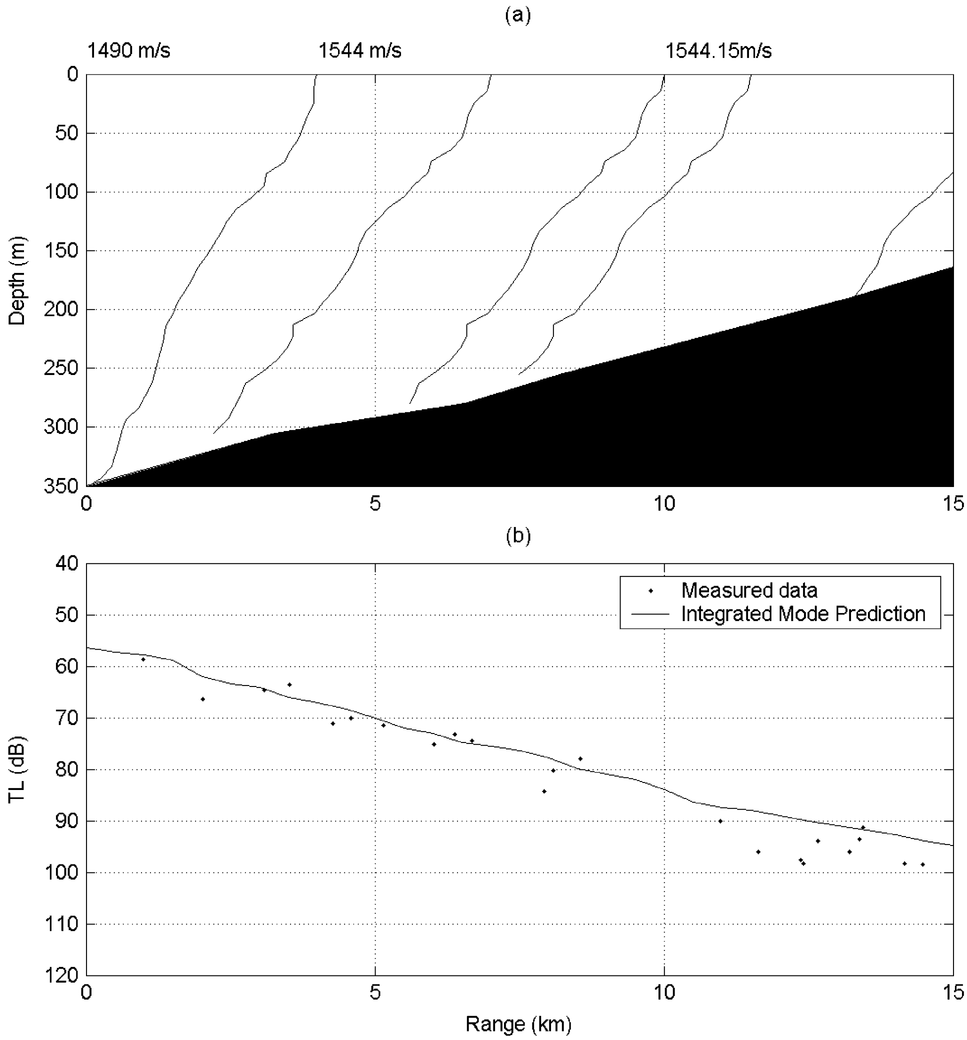

Fig. 6. Example Case D: TL versus range for a 142-Hz source in an environment with horizontally variable bottom depth and sound speed.

local wavenumber at source and receiver. The Fresnel integral is also computed as in the original derivation, i.e., by substitution of (31) in the case of infinite integration limits, or by an approximation based on integration by parts. The remaining factors involving the (range dependent) accumulation of the phase functions along the path Γ, and the horizontal wavenumber associated with the stationary phase point k_0^* comprise the changes required to extend the solution to horizontally variable media.

V. Discussion of Results

The following provides samples of calculations made with the Integrated Mode model described herein, with comparisons to predictions computed using other models and measured results. In each case, measured results or other model calculations are shown as dots, and the prediction of the Integrated Mode model is represented by a solid line.

A. Surface Duct Propagation

The first sample case involves propagation of a 3520-Hz signal in a surface duct [14]. The field is dominated by a small number of low order modes and both measurements and model calculations are restricted to propagation within the duct. The source and receiver are in the duct at depths of 20 and 50 feet, respectively, and the wind speed is 5 knots. The upper 1000 feet of the SSP and the results for Sample Case A are shown in Fig. 3(a) and (b), respectively.

TABLE III
Bottom Sediment Parameters for Example Case E

Sthick	1.32 sec
Xratio	0.999
Xgrad	1.3
Beta	-0.5
Satten	0
Sagrad	0.0001
Sdense	1.44
Xdense	1.63
Xthick	0.65
Baslos	0.45

B. Deep Water Convergence Zone Propagation

The second sample case involves propagation of a 350-Hz signal in a convergence zone environment [15]. The source, receiver, and bottom depths are 400, 600, and 16 920 feet, respectively, and the wind speed is 4 knots. The bottom was mod-

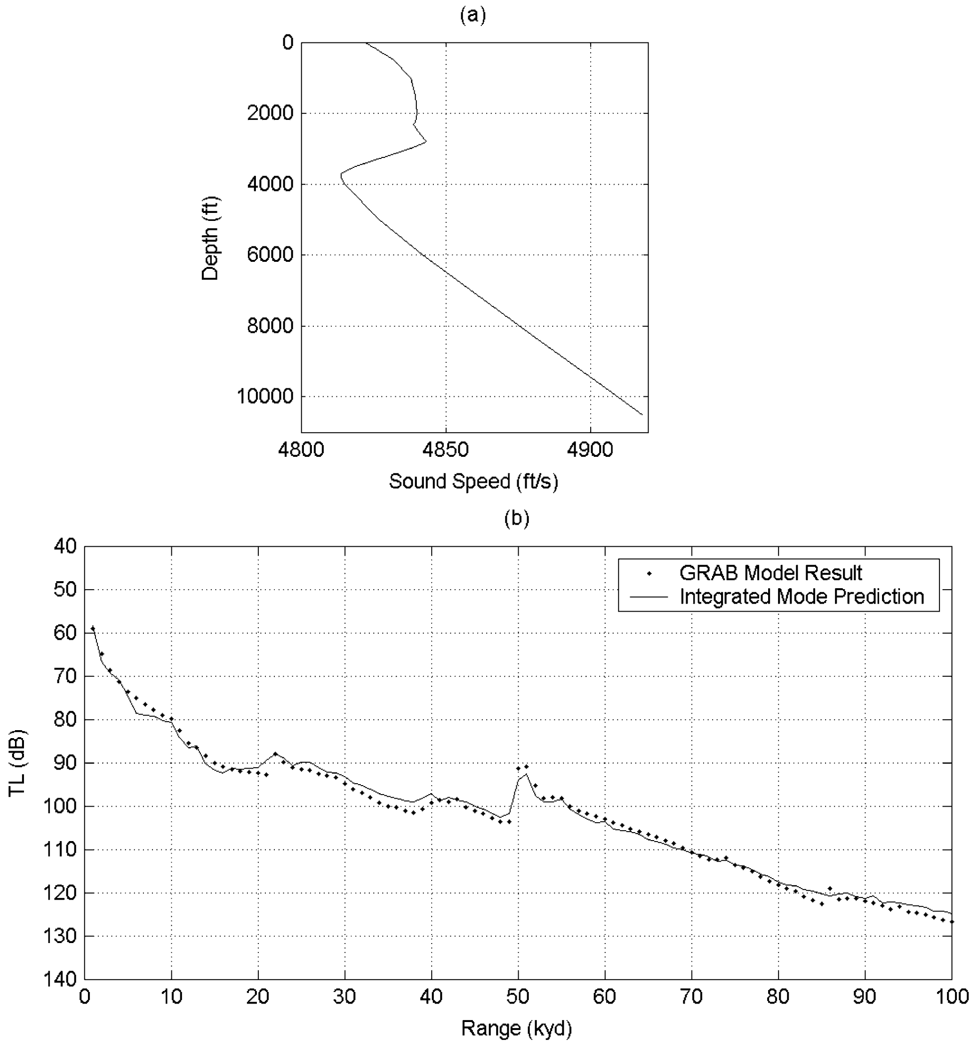

Fig. 7. Sample Case E: TL versus range for a 3500-Hz source in the Norwegian Sea.

eled using a plane wave reflection coefficient with the loss per bounce as shown with the SSP and the results, in Fig. 4(a), (b), and (c), respectively.

C. Shallow Water Propagation

The third sample case represents propagation of a 3860-Hz signal in a shallow water environment with a single downward-refracting SSP and highly variable downslope bathymetry [16]. The source and receiver were both at a depth of 25 feet. The bottom was modeled using a plane wave reflection coefficient with the loss per bounce as shown in Fig. 5(a). The bathymetry profile for Sample case C and an overlay of the sound speed variation with depth are shown in Fig. 5(b), and the results for Sample Case C are presented in Fig. 5(c).

D. Propagation in an Environment With Horizontally Variable Sound Speed, Bottom Depth, and Bottom Type

The fourth sample case represents propagation of a 142-Hz signal in a shallow water environment with horizontally variable sound speed, bottom type, and bathymetry [17]. The SSPs at ranges of 0, 3, 6, 7.5, and 12.09 km are plotted with the upslope bathymetry for Sample case D in Fig. 6(a). The bottom sediment parameters by range are given in Table II, where Sthick is sediment thickness represented by two-way travel time through the sediment in seconds, Xratio is the ratio of sediment to water column sound speed at the sediment interface, Xgrad is the initial sediment sound speed gradient in 1/s, Beta is a sediment SSP curvature parameter, Satten is initial sediment attenuation in dB/m/kHz, Sagrad is sediment attenuation gradient in dB/m/kHz/m, Sdense is sediment density in g/cc, Xdense and Xthick are the density and thickness of a hypothetical thin layer at the sediment/water column interface in g/cc and m, respectively, and Baslos is the basement reflection coefficient. The source and receiver are both at a depth of 18 meters.

The results for Sample Case D are presented in Fig. 6(b).

E. Comparison to Navy Standard Grab Model

The fifth sample case represents propagation of a 3500-Hz signal in the Norwegian Sea in winter with results compared to those of the Navy Standard Gaussian RAy Bundle (GRAB) model of Weinberg [5]. The bottom depth is 10 500 feet and the source and receiver are at depths of 325 and 850 feet, respectively. Bottom parameters are given in Table III with parameters and units as defined in Sample Case D, and the wind speed is 16 knots. The SSP and results are shown in Fig. 7(a) and (b), respectively.

VI. Summary

The multipath expansion method of solving the Helmholtz wave equation to describe the underwater sound field for a fixed point source in a plane multilayered medium has been discussed. This approach has been extended to account for horizontal variations in bottom depth, bottom type, and sound speed in the stationary phase approximation. A comparison of calculations with both measured data and results of the Navy standard model covering a range of frequencies and environmental variations has been presented.

Acknowledgment

Portions of this paper are included in a thesis submitted in partial fulfillment of the requirements of the M.S. degree in electrical engineering at the University of Rhode Island, Kingston, 2001.

References

[1] G. A. Leibiger, "Wave propagation in an inhomogeneous medium with slow spatial variation," Ph.D. dissertation, Stevens Institute of Technol., Hoboken, NJ, 1968.

[2] Oceanographic and atmospheric master library summary, Naval Oceanographic Office, Systems Integration Division, Stennis Space Center, MS, 1984. The original Leibiger model was adopted as "raymode".

[3] H. Weinberg and R. Burridge, "Horizontal ray theory for ocean acoustics," *J. Acoust. Soc. Amer.*, vol. 55, no. 1, 1974.

[4] H. Weinberg, "Application of ray theory to acoustic propagation in horizontally stratified oceans," *J. Acoust. Soc. Amer.*, vol. 58, no. 1, 1975.

[5] H. Weinberg and R. Keenan, "Gaussian ray bundles for modeling high-frequency propagation loss under shallow-water conditions," *J. Acoust. Soc. Amer.*, vol. 100, no. 3, 1996.

[6] *Physics of Sound in the Sea*: National Defense Research Committee, Peninsula, 1988.

[7] I. Tolstoy and C. S. Clay, *Ocean Acoustics*. New York: McGraw-Hill, 1966.

[8] C. S. Clay and H. Medwin, *Acoustical Oceanography: Principles and Applications*. New York: Wiley, 1977.

[9] F. B. Jensen, H. Schmidt, W. Kuperman, and M. Porter, *Computational Ocean Acoustics*: Amer. Inst. Phys., 1994.

[10] E. T. Copson, *An Introduction to the Theory of Functions of a Complex Variable*, U.K.: Oxford Univ., 1960.

[11] B. Friedman, *Principles and Techniques of Applied Mathematics*. New York: Dover, 1990.

[12] I. Stakgold, *Green's Functions and Boundary Value Problems*. New York: Wiley, 1979.

[13] H. S. Wilf, *Mathematics for the Physical Sciences*. New York: Dover, 1962.

[14] K. V. Kovolenko, One way propagation loss measurements at AUTEC range, 1973. Unpublished data.

[15] C. A. Clark, "CAAM Model Validation: Active Raymode," Sonalysts, Inc., CA1-0501-037, 1991.

[16] Transmission loss and oceanographic measurements collected during ship antisubmarine warfare readiness/effectiveness measuring (SHAREM) exercises, Naval Oceanographic Office, 1997.

[17] D. King, Naval Research Laboratory, Acoustics Division Code 7180, Stennis Space Center, MS 39529, Sep. 1997. (private communication).

Cathy Ann Clark (S'03–M'04) received the B.S. degree in mathematics from Morehead State University, Morehead, KY, and the M.S. degree in mathematics from the University of Connecticut. She received another M.S. degree in electrical engineering and the Ph.D. degree in applied mathematics with an emphasis on nonlinear rational difference equations, both from the University of Rhode Island.

She was with the Western Union Corporation, Mahwah, NJ, from 1969 to 1973 as a communications systems analyst. From 1973 to 1979 she was a lecturer in mathematics with the University of Connecticut and various community colleges in Connecticut. From 1979 to 1997 she was with Sonalysts, Inc., in Waterford, CT, working on underwater sound propagation modeling, satellite communications, and submarine tactical decision aid development. Since 1997 she has been with the Submarine Sonar Department, Naval Undersea Warfare Center, Newport, RI. Her research interests include underwater acoustics, boundary value problems, and difference equations.

Dr. Clark is a member of the American Mathematical Society, the Society for Industrial and Applied Mathematics, and the International Society of Difference Equations.